Self-duality relation for a three-dimensional spin-$^1/_2$ lattice system

1985 J. Phys. A: Math. Gen. 18 1579

(http://iopscience.iop.org/0305-4470/18/9/037)

J. Phys. A: Math. Gen. **18** (1985) 1579-1582. Printed in Great Britain

COMMENT

Self-duality relation for a three-dimensional spin-$\frac{1}{2}$ lattice system

Héctor J Giacomini

Instituto de Física de Rosario, IFIR, CONICET, UNR, Pellegrini 250, 2000-Rosario, Argentina

Received 24 October 1984

Abstract. A three-dimensional system consisting of a set of interacting two-dimensional Ising models is introduced. It is shown that this model is self-dual. It contains the three-dimensional Ising model as a particular case.

Duality transformations have several roles in statistical mechanics. One role is to relate the properties of one system at high temperature to those of another (or the same) system at low temperature. If the system happens to be self-dual, one can, in some cases, determine its critical points and have improved control over approximate and numerical studies.

Another role is a rigorous shift of ground from one description of a system to a very different description, which is either known, easier to interpret (Peskin 1978, Savit 1980), or more amenable to treatment. In general, all ordinary spin systems in three dimensions have been shown to be dual to gauge systems with a local symmetry. Very few models are self-dual in three dimensions. For spin-$\frac{1}{2}$ systems the only relevant case is the gauge invariant Ising model with external field.

In this comment we study a three-dimensional model consisting of a set of interacting two-dimensional Ising models, and we show that it is self-dual.

The partition function of the model is

$$Z(J_1,\ldots,J_6)=\sum_{\{s_x^{(1)}\}}\sum_{\{s_x^{(2)}\}}\exp\Big(\sum_x J_1 s_x^{(1)}s_{x+\mu_1}^{(1)}+J_2 s_x^{(1)}s_{x+\mu_3}^{(1)}+J_3 s_x^{(2)}s_{x+\mu_2}^{(2)}+J_4 s_x^{(2)}s_{x+\mu_3}^{(2)}$$
$$+J_5 s_x^{(1)}s_{x+\mu_1}^{(1)}s_x^{(2)}s_{x+\mu_2}^{(2)}+J_6 s_x^{(1)}s_{x+\mu_3}^{(1)}s_x^{(2)}s_{x+\mu_3}^{(2)}\Big) \tag{1}$$

where x indicates a generic point of the lattice, μ_1, μ_2 and μ_3 are unit vectors in the x, y and z directions, respectively; and $s_x^{(1)}$ and $s_x^{(2)}$ are Ising variables. We assume periodic boundary conditions on the cubic lattice. When $J_5=0$ and $J_6=0$ the model decouples in a set of $2N^{1/3}$ independent two-dimensional Ising models, where N is the number of lattice sites. Half of these 2D Ising models are sitting in planes parallel to the xz plane and the other half in planes parallel to the yz plane.

This system bears resemblance to the 2D Ashkin-Teller model, which consists, when it is expressed in terms of Ising variables (Fan 1972), of two interacting 2D Ising models defined on the same plane. When $J_6=\infty$ and $J_5=0$ the constraint results:

$$s_x^{(1)}s_{x+\mu_3}^{(1)}s_x^{(2)}s_{x+\mu_3}^{(2)}=1, \tag{2}$$

0305-4470/85/091579+04\$02.25

therefore $s_x^{(1)} = s_x^{(2)}$, and in consequence the system reduces to the 3D anisotropic Ising model without external field. When $J_1 = J_3 = J_6 = 0$ the 3D anisotropic gauge invariant Ising model evaluated in the axial gauge results. The variable $s_x^{(1)}$ must be identified with the links in the y direction and $s_x^{(2)}$ with those in the x direction. The links in the z direction are equal to one in the axial gauge. The Hamiltonian of our model has several symmetries. It remains invariant to the reversal of all spins $s_x^{(1)}$ or all spins $s_x^{(2)}$. Moreover, it remains invariant to the reversal of all spin $s_x^{(1)}$ on any of the planes parallel to the xz plane, and to the reversal of all spins $s_x^{(2)}$ on any of the planes parallel to the yz plane.

We will establish the self-duality of this model, by using a procedure introduced in a previous paper (Giacomini 1984a). By linearising (1) in the spin variables and introducing six variables, which take the values zero and one, in order to expand the resulting products, we obtain

$$Z(J_1, \dots, J_6) = [\cosh(J_1) \dots \cosh(J_6)]^N \sum_{\{s_x^{(1)}\}} \sum_{\{s_x^{(2)}\}} \sum_{\{n_x^{(1)}\}} \dots \sum_{\{n_x^{(6)}\}}$$
$$\times \prod_x [\alpha_1^{n_x^{(1)}} \dots \alpha_6^{n_x^{(6)}} (s_x^{(1)})^{n_x^{(1)} + n_{x-\mu_1}^{(1)} + n_x^{(2)} + n_{x-\mu_3}^{(2)} + n_x^{(5)} + n_{x-\mu_1}^{(5)} + n_x^{(6)} + n_{x-\mu_3}^{(6)}}$$
$$\times (s_x^{(2)})^{n_x^{(3)} + n_{x-\mu_2}^{(3)} + n_x^{(4)} + n_{x-\mu_3}^{(4)} + n_x^{(5)} + n_{x-\mu_2}^{(5)} + n_x^{(6)} + n_{x-\mu_3}^{(6)}}] \tag{3}$$

where $\alpha_i = \tanh(J_i)$.

The variables $s_x^{(1)}$ and $s_x^{(2)}$ are decoupled in (3). Summing them up leads to

$$2(J_1, \dots, J_6) = 2^{2N} [\cosh(J_1) \dots \cosh(J_6)]^N \sum_{\{n_x^{(1)}\}} \dots \sum_{\{n_x^{(6)}\}}$$
$$\times \prod_x [\alpha_1^{n_x^{(1)}} \dots \alpha_6^{n_x^{(6)}} \delta_2(n_x^{(1)} + n_{x-\mu_1}^{(1)} + n_x^{(2)} + n_{x-\mu_3}^{(2)} + n_x^{(5)} + n_{x-\mu_1}^{(5)} + n_x^{(6)}$$
$$+ n_{x-\mu_3}^{(6)}) \delta_2(n_x^{(3)} + n_{x-\mu_2}^{(3)} + n_x^{(4)} + n_{x-\mu_3}^{(4)} + n_x^{(5)} + n_{x-\mu_2}^{(5)} + n_x^{(6)} + n_{x-\mu_3}^{(6)})] \tag{4}$$

where $\delta_2(n)$ is a Kronecker delta function modulo two, it is zero if n is odd and one if n is even.

By introducing Ising variables according to $s_x^{(i)} = 2n_x^{(i)} - 1$ and taking into account the identities

$$\delta_2(n_x^{(1)} + n_{x-\mu_1}^{(1)} + n_x^{(2)} + n_{x-\mu_3}^{(2)} + n_x^{(5)} + n_{x-\mu_1}^{(5)} + n_x^{(6)} + n_{x-\mu_3}^{(6)})$$
$$= \tfrac{1}{2}(1 + s_x^{(1)} s_{x-\mu_1}^{(1)} s_x^{(2)} s_{x-\mu_3}^{(2)} s_x^{(5)} s_{x-\mu_1}^{(5)} s_x^{(6)} s_{x-\mu_3}^{(6)})$$
$$\delta_2(n_x^{(3)} + n_{x-\mu_2}^{(3)} + n_x^{(4)} + n_{x-\mu_3}^{(4)} + n_x^{(5)} + n_{x-\mu_2}^{(5)} + n_x^{(6)} + n_{x-\mu_3}^{(6)})$$
$$= \tfrac{1}{2}(1 + s_x^{(3)} s_{x-\mu_2}^{(3)} s_x^{(4)} s_{x-\mu_3}^{(4)} s_x^{(5)} s_{x-\mu_2}^{(5)} s_x^{(6)} s_{x-\mu_3}^{(6)}), \tag{5}$$

we obtain

$$Z(J_1, \dots, J_6) = 2^{-6N} \exp[N(J_1 + \dots + J_6)] \sum_{\{s_x^{(1)}\}} \dots \sum_{s_x^{(6)}\}}$$
$$\times \prod_x (1 + v_1 s_x^{(1)})(1 + v_2 s_x^{(2)})(1 + v_3 s_x^{(3)})(1 + v_4 s_x^{(4)})(1 + v_5 s_x^{(5)})$$
$$\times (1 + v_6 s_x^{(6)})(1 + s_x^{(1)} s_{x+\mu_1}^{(1)} s_x^{(2)} s_{x+\mu_3}^{(2)} s_x^{(5)} s_{x+\mu_1}^{(5)} s_x^{(6)} s_{x+\mu_3}^{(6)})$$
$$\times (1 + s_x^{(3)} s_{x+\mu_2}^{(3)} s_x^{(4)} s_{x+\mu_3}^{(4)} s_x^{(5)} s_{x+\mu_2}^{(5)} s_x^{(6)} s_{x+\mu_3}^{(6)}) \tag{6}$$

where $v_i = e^{-2J_i}$, and we have changed the signs of the unit vectors based on the symmetry of the lattice. With the aim of simplifying the resulting constraints we make the following change of variables

$$s_x^{(1)} \to s_x^{(1)} s_x^{(5)}, \qquad s_x^{(2)} \to s_x^{(2)} s_x^{(6)},$$
$$s_x^{(3)} \to s_x^{(3)} s_x^{(5)}, \qquad s_x^{(4)} \to s_x^{(4)} s_x^{(6)}. \tag{7}$$

Therefore (6) gives

$$Z(J_1, \dots, J_6) = 2^{-6N} \exp[N(J_1 + \dots + J_6)] \sum_{\{s_x^{(1)}\}} \dots \sum_{\{s_x^{(6)}\}}$$
$$\times \prod_x (1 + v_1 s_x^{(1)} s_x^{(5)})(1 + v_2 s_x^{(2)} s_x^{(6)})(1 + v_3 s_x^{(3)} s_x^{(5)})(1 + v_4 s_x^{(4)} s_x^{(6)})$$
$$\times (1 + v_5 s_x^{(5)})(1 + v_6 s_x^{(6)})(1 + s_x^{(1)} s_{x+\mu_1}^{(1)} s_x^{(2)} s_{x+\mu_3}^{(2)})$$
$$\times (1 + s_x^{(3)} s_{x+\mu_2}^{(3)} s_x^{(4)} s_{x+\mu_3}^{(4)}). \tag{8}$$

Now, the constraints are

$$s_x^{(1)} s_{x+\mu_1}^{(1)} s_x^{(2)} s_{x+\mu_3}^{(2)} = 1, \qquad s_x^{(3)} s_{x+\mu_2}^{(3)} s_x^{(4)} s_{x+\mu_3}^{(4)} = 1. \tag{9}$$

The general solution of (9) is (Balian *et al* 1975)

$$s_x^{(1)} = s_x^{(7)} s_{x+\mu_3}^{(7)}, \qquad s_x^{(3)} = s_x^{(8)} s_{x+\mu_3}^{(8)},$$
$$s_x^{(2)} = s_x^{(7)} s_{x+\mu_1}^{(7)}, \qquad s_x^{(4)} = s_x^{(8)} s_{x+\mu_2}^{(8)} \tag{10}$$

where $s_x^{(7)}$ and $s_x^{(8)}$ are Ising variables too. The variables $s_x^{(5)}$ and $s_x^{(6)}$ are decoupled in (8). Summing them up, and taking into account (10), we get

$$Z(J_1, \dots, J_6) = 2^{-2N} \exp[N(J_1 + \dots + J_6)] \sum_{\{s_x^{(7)}\}} \sum_{\{s_x^{(8)}\}}$$
$$\times \prod_x (1 + v_1 v_3 s_x^{(7)} s_{x+\mu_3}^{(7)} s_x^{(8)} s_{x+\mu_3}^{(8)} + v_1 v_5 s_x^{(7)} s_{x+\mu_3}^{(7)}$$
$$+ v_3 v_5 s_x^{(8)} s_{x+\mu_3}^{(8)})(1 + v_2 v_6 s_x^{(7)} s_{x+\mu_1}^{(7)}$$
$$+ v_2 v_4 s_x^{(7)} s_{x+\mu_1}^{(7)} s_x^{(8)} s_{x+\mu_2}^{(8)} + v_6 v_4 s_x^{(8)} s_{x+\mu_2}^{(8)}). \tag{11}$$

Now, this expression can be written as

$$Z(J_1, \dots, J_6) = \frac{\exp[N(J_1 + \dots + J_6)]}{[4(1 + \gamma_2\gamma_4\gamma_6)(1 + \gamma_1\gamma_3\gamma_5)]^N} \sum_{\{s_x^{(1)}\}} \sum_{\{s_x^{(2)}\}}$$
$$\times \prod_x (1 + \gamma_2 s_x^{(1)} s_{x+\mu_3}^{(1)})(1 + \gamma_4 s_x^{(2)} s_{x+\mu_3}^{(2)})$$
$$\times (1 + \gamma_6 s_x^{(1)} s_{x+\mu_3}^{(1)} s_x^{(2)} s_{x+\mu_3}^{(2)})(1 + \gamma_1 s_x^{(1)} s_{x+\mu_1}^{(1)})$$
$$\times (1 + \gamma_3 s_x^{(2)} s_{x+\mu_2}^{(2)})(1 + \gamma_5 s_x^{(1)} s_{x+\mu_1}^{(1)} s_x^{(2)} s_{x+\mu_2}^{(2)}) \tag{12}$$

where we have changed the name of the variables, and the parameters γ_i are determined by the following equations

$$\frac{\gamma_2 + \gamma_4\gamma_6}{1 + \gamma_2\gamma_4\gamma_6} = v_1 v_5 \qquad \frac{\gamma_4 + \gamma_2\gamma_6}{1 + \gamma_2\gamma_4\gamma_6} = v_3 v_5$$
$$\frac{\gamma_6 + \gamma_2\gamma_4}{1 + \gamma_2\gamma_4\gamma_6} = v_1 v_3 \qquad \frac{\gamma_1 + \gamma_3\gamma_5}{1 + \gamma_1\gamma_3\gamma_5} = v_2 v_6 \tag{13}$$
$$\frac{\gamma_3 + \gamma_1\gamma_5}{1 + \gamma_1\gamma_3\gamma_5} = v_4 v_6 \qquad \frac{\gamma_5 + \gamma_1\gamma_3}{1 + \gamma_1\gamma_3\gamma_5} = v_2 v_4.$$

Finally, by defining the dual couplings J_i^* through the relation

$$\tanh(J_i^*) = \gamma_i \tag{14}$$

(12) becomes

$$Z(J_1, \ldots, J_6) = \frac{\exp[N(J_1 + \ldots + J_6)]}{[4(1+\gamma_2\gamma_4\gamma_6)(1+\gamma_1\gamma_3\gamma_5)\cosh(J_1^*)\ldots\cosh(J_6^*)]^N}$$

$$\times \sum_{\{s_x^{(1)}\}} \sum_{\{s_x^{(2)}\}} \exp\Big(\sum_x J_1^* s_x^{(1)} s_{x+\mu_1}^{(1)} + J_2^* s_x^{(1)} s_{x+\mu_3}^{(1)} + J_3^* s_x^{(2)} s_{x+\mu_2}^{(2)}$$

$$+ J_4^* s_x^{(2)} s_{x+\mu_3}^{(2)} + J_5^* s_x^{(1)} s_{x+\mu_1}^{(1)} s_x^{(2)} s_{x+\mu_2}^{(2)} + J_6^* s_x^{(1)} s_{x+\mu_3}^{(1)} s_x^{(2)} s_{x+\mu_3}^{(2)}\Big). \tag{15}$$

Therefore, the self-duality relation is expressed as

$$Z(J_1, \ldots, J_6) = \left(\frac{\exp(J_1 + \ldots + J_6)}{4(1+\gamma_2\gamma_4\gamma_6)(1+\gamma_1\gamma_3\gamma_5)\cosh(J_1^*)\ldots\cosh(J_6^*)}\right)^N Z(J_1^*, \ldots, J_6^*). \tag{16}$$

When $J_3 = J_4 = J_5 = J_6 = 0$, (16) reduces to the self-duality relation of the anisotropic two-dimensional Ising model. For the case $J_6 = \infty$ and $J_5 = 0$ (16) gives the duality relation between the 3D Ising model and its gauge invariant version (Balian *et al* 1975).

The self-duality relation (16) is the same as would be obtained for the anisotropic Ashkin–Teller model with a Hamiltonian given by

$$H = -\sum_x (J_1 s_x^{(1)} s_{x+\mu_1}^{(1)} + J_2 s_x^{(1)} s_{x+\mu_2}^{(1)} + J_3 s_x^{(2)} s_{x+\mu_1}^{(2)} + J_4 s_x^{(2)} s_{x+\mu_2}^{(2)}$$

$$+ J_5 s_x^{(1)} s_{x+\mu_1}^{(1)} s_x^{(2)} s_{x+\mu_1}^{(2)} + J_6 s_x^{(1)} s_{x+\mu_2}^{(1)} s_x^{(2)} s_{x+\mu_2}^{(2)}) \tag{17}$$

where $s_x^{(1)}$ and $s_x^{(2)}$ are Ising variables and μ_1 and μ_2 are unit vectors in the horizontal and vertical directions, respectively. The similarities with the 2D Ashkin–Teller model induce us to think that the model (1) could present a non-universal behaviour (Barber 1980). We believe that this aspect deserves further analysis. At last, we can say that the self-duality of this model is a consequence of the fact that it is constructed by an adequate combination of self-dual models (2D Ising models in this case), as is explained in a previous paper (Giacomini 1985b).

References

Balian R, Drouffe J M and Itzykson C 1975 *Phys. Rev.* D **11** 2098
Barber M 1980 *Phys. Rep.* **59** 375
Fan C 1972 *Phys. Lett.* **39A** 136
Giacomini H 1985a *J. Phys. A: Math. Gen.* **18** 1499
—— 1985b *J. Phys. A: Math. Gen.* **18** 1505
Peskin M 1978 *Ann. Phys., NY* **113** 122
Savit R 1980 *Rev. Mod. Phys.* **52** 453